MULTIPLICATIVE C-FREE CONVOLUTION

MIHAI POPA AND JIUN-CHAU WANG

Abstract. Using the combinatorics of non-crossing partitions, we construct a conditionally free analogue of the Voiculescu's S-transform. The result is applied to analytical description of conditionally free multiplicative convolution and characterization of infinite divisibility.

AMS Subject Classification: 46L53; 05A18, 60E07

Keywords: conditionally free independence, multiplicative convolution, S-transform, infinite divisibility, non-crossing and non-crossing linked partitions

1. Introduction

The paper presents some results in conditionally free (or, shorter, following [8], c-free) probability. The concept of c-freeness was developed in early '90's (see [6], [7],[8]) as a particular situation of freeness. Namely, if $\mathfrak{A}$ is an algebra and $\varphi, \psi : \mathfrak{A} \longrightarrow \mathbb{C}$ are two normalized linear functionals, then the family $\{\mathfrak{A}_j\}_j$ of subalgebras of $\mathfrak{A}$ is said to be c-free if

  (i) $\psi(a_1 \cdots a_n) = 0$
  (ii) $\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n)$

for all $a_j \in \mathfrak{A}_{\varepsilon(j)}$ and $\varepsilon(j) \in \{1, 2\}$, such that $\varepsilon(1) \neq \varepsilon(2) \neq \cdots \neq \varepsilon(n)$ and $\psi(a_j) = 0$, $1 \leq j \leq n$.

Two important tools in free probability theory are the R and S transforms. Those are power series with the property that if X and Y are free random variables, then $R_{X+Y} = R_X + R_Y$ and $S_{XY} = S_X S_Y$. While a c-free version of the R-transform is constructed in [8] and used in several papers (such as [21], [16], [14]) for the study of c-free additive convolution of measures or of c-freeness with amalgamation, the literature lacked a similar development for the multiplicative case.

The present paper shows the construction of a suitable c-free version of the S-transform (in fact, as in [9], a more natural choice is its inverse, called the T-transform), and demonstrates some of its applications in limit theorems and characterization of infinite divisibility.

The material is structured in seven sections. The second section contains notations and preliminary results used throughout the paper, but mostly in the third section. The third section presents the construction of the ^cT-transform and proves its multiplicative property. The argument is based on enumerative combinatorial techniques, in the spirit of [15] - the lack of a Fock space model makes difficult an analytical procedure, as in [12]. The forth section defines the multiplicative c-free convolution of two measures on the unit circle and presents its connections to free

and boolean multiplicative convolutions. The fifth and the sixth sections present applications of the multiplicative property of the ${}^cT$-transform for the study of limit distributions, respectively for the characterization of infinite divisibility. These sections are using more analytical techniques, analogue to [3], [4] and [5]. The last section - the appendix - is showing an explicit combinatorial formula for computing the coefficients of the T- and ${}^cT$-transforms. The computations, generalizing a result from [9], are demonstrating the use of non-crossing linked partitions in free probability.

2. PRELIMINARIES

2.1. **The R, T and ${}^cR$ transforms.** Let $\mathfrak{A}$ be a complex unital algebra endowed with two linear functionals, ψ and φ. For $a_1, \ldots, a_n \in \mathfrak{A}$, the free cumulant $R^n(a_1, \ldots, a_n)$, respectively the c-free cumulant ${}^cR^n(a_1, \ldots, a_n)$, are defined by the recurrences:

$$\psi(a_1 \cdots a_n) = \sum_{p=1}^{n} \sum R^p(a_{i(1)} \cdots a_{i(p)}) \left[\prod_{k=1}^{p-1} \psi(a_{i(k)+1} \cdots a_{i(k+1)-1}) \right] \psi(a_{i(p)+1} \cdots a_n)$$

$$\varphi(a_1 \cdots a_n) = \sum_{p=1}^{n} \sum {}^cR^p(a_{i(1)} \cdots a_{i(p)}) \left[\prod_{k=1}^{p-1} \psi(a_{i(k)+1} \cdots a_{i(k+1)-1}) \right] \varphi(a_{i(p)+1} \cdots a_n)$$

where in both lines the second summation is done over all $1 = i(0) < i(1) < \cdots < i(k) \le n$.

For $X \in \mathfrak{A}$, we will write R_X^n and ${}^cR_X^n$ for $R^n(X, \ldots, X)$, respectively ${}^cR^n(X, \ldots, X)$. The R-, respectively ${}^cR$-transform of X are the formal power series

$$R_X(z) = \sum_{n=1}^{\infty} r_X^n z^n, \qquad {}^cR_X(z) = \sum_{n=1}^{\infty} {}^cR_X^n z^n$$

Let $m_X(z)$, respectively $M_X(z)$ be the moment-generating series of X with respect to ψ and φ, i. e. $m_X(z) = \sum_{n=1}^{n} \psi(X^n) z^n$ and $M_X(z) = \sum_{n=1}^{n} \varphi(X^n) z^n$. As shown in [15] and [8], the definitions of R^n and ${}^cR^n$ give

$$(1) \qquad\qquad\qquad R\left(z[1 + m_X(z)]\right) = m_X(z)$$

$$(2) \qquad\qquad {}^cR\left(z[1 + m_X(z)]\right)(1 + M_X(z)) = M_X(z)(1 + m_X(z))$$

Two elements, X and Y, from $\mathfrak{A}$ are said to be conditionally free (c-free) in $(\mathfrak{A}, \varphi, \psi)$ if the subalgebras generated by them in $\mathfrak{A}$ are conditionally free, as defined in Introduction. The key properties of the R and ${}^cR$ transforms are summarized in the following result:

Lemma 2.1. *Let X, Y be c-free in ${}^cR^n(X, \ldots, X)$.*

(i) *Let $a_k \in \{X, Y\}$, $1 \le k \le n$. If there exist $k \ne l$ such that $a_k \ne a_l$, then $R^n(a_1, \ldots, a_n) = 0$ and ${}^cR^n(a_1, \ldots, a_n) = 0$*

(ii) *Let T_X be the formal power series defined by $T_X(z) = \left(\frac{1}{z} R(z)\right) \circ \left(R^{\langle -1 \rangle}(z)\right)$, where $(F(z))^{\langle -1 \rangle}$ is the substitutional inverse of the formal series $F(z)$. Then*

$$T_{XY}(z) = T_X(z) T_Y(z).$$

Proofs for the first part of (i) can be found, for example, in [12], Theorem 2.2, and [15], Theorem 16.16. Second part of (i) is proved in [8], Theorem 3.1.

For (ii), in [12], Corollary 2.4. or Theorem 3.5, and [15], Corollary 18.17, the property is proved for $S(z) = [T(z)]^{-1}$ (inverse with respect to multiplication). As in [9], for the purpose of this paper it will be simpler to consider $T(z) = [S(z)]^{-1}$.

2.2. Boolean independence, the η- and B-transforms.

Let now $\mathfrak{A}$ be a unital algebra and $\varphi : \mathfrak{A} \longrightarrow \mathbb{C}$ be a normalized functional. Two subalgebras $\mathfrak{A}_1, \mathfrak{A}_2$ are said to be *boolean independent* if

$$\varphi(x_1 y_1 x_2 y_2 \cdots) = \varphi(x_1)\varphi(y_1)\cdots,$$

for all $x_k \in \mathfrak{A}_1$ and $y_k \in \mathfrak{A}_2$. We will say that $X, Y \in \mathfrak{A}$ are boolean independent if the nonunital algebras generated in $\mathfrak{A}$ by X, respectively Y are boolean independent.

If $\mathfrak{A} = \mathbb{C} \oplus \mathfrak{A}_0$ (direct sum of vector spaces), and $\psi = Id_\mathbb{C} \oplus 0_{\mathfrak{A}_0}$, then conditional freeness with respect to (φ, ψ) is equivalent to boolean independence with respect to φ.

For the results in Section 4-6, we need the definitions and results below (see [18] or [11] for proves).

Definition 2.2. *Let $X \in \mathfrak{A}$ and $M_X(z)$ be the moment generating series of X with respect to φ, that is $M_X(z) = \sum_{n=1}^{\infty} \varphi(X^n) z^n$. The η-, respectively B-transforms of X are the formal power series given by the relations $\eta_X(z) = \frac{M_X(z)}{1+M_X(z)}$, respectively $B_X(z) = \frac{1}{z}\eta_X(z)$.*

Proposition 2.3. *If X and Y are two boolean independent elements of $\mathfrak{A}$, then:*

(i) $\eta_{X+Y}(z) = \eta_X(z) + \eta_Y(z)$.
(ii) $B_{X+Y}(z) = B_X(z) + B_Y(z)$ and $B_{(1+X)(1+Y)}(z) = B_{(1+X)}(z) \cdot B_{(1+Y)}(z)$.

2.3. Non-crossing partitions.

By a partition on the ordered set $\langle n \rangle = \{1, 2, \ldots, n\}$ we will understand a collection of mutually disjoint subsets of $\langle n \rangle$, $\gamma = (B_1, \ldots, B_q)$, called *blocks* whose union is the entire set $\langle n \rangle$. A *crossing* is a sequence $i < j < k < l$ from $\langle n \rangle$ with the property that there exist two different blocks B_r and B_s such that $i, k \in B_r$ and $j, l \in B_s$. A partition that has no crossings will be called non-crossing. The set of all non-crossing partitions on $\langle n \rangle$ will be denoted by $NC(n)$.

For $\gamma \in NC(n)$, a block $B = (i_1, \ldots, i_k)$ of γ will be called *interior* if there exists another block $D \in \gamma$ and $i, j \in D$ such that $i < i_1, i_2, \ldots, i_k < j$. A block will be called *exterior* if is not interior. The set of all interior, respectively exterior blocks of γ will be denoted by $Int(\gamma)$, respectively $Ext(\gamma)$.

With the above notations, the recurrences defining R^n, respectively $^c R^n$ can be written as

$$\psi(a_1 \cdots a_n) = \sum_{\substack{\gamma \in NC(n)}} \prod_{\substack{B \in \gamma \\ B = (i_1, \ldots, i_k)}} R^k(a_{i_1}, \ldots, a_{i_k})$$

$$\varphi(a_1 \cdots a_n) = \sum_{\substack{\gamma \in NC(n)}} \prod_{\substack{D \in Ext(\gamma) \\ D = (i_1, \ldots, i_k)}} {}^c R^k(a_{i_1}, \ldots, a_{i_k}) \prod_{\substack{B \in Int(\gamma) \\ B = (j_1, \ldots, j_s)}} R^s(a_{j_1}, \ldots, a_{j_s})$$

$NC(n)$ has a lattice structure with respect to the reversed refinement order, with the biggest, respectively smallest element $\mathbb{1}_n = (1, 2, \ldots, n)$, respectively $0_n = (1), \ldots, (n)$. For $\pi, \sigma \in NC(n)$ we will denote by $\pi \bigvee \sigma$ their join (smallest common upper bound).

The Kreweras complementary $Kr(\pi)$ of $\pi \in NC(n)$ is defined as follows. Consider the symbols $\overline{1}, \ldots, \overline{n}$ such that $1 < \overline{1} < 2 < \cdots < n < \overline{n}$. Then $Kr(\pi)$ is the biggest element of $NC(\overline{1}, \ldots, \overline{n}) \cong NC(n)$ such that

$$\pi \cup Kr(\pi) \in NC(1, \overline{1}, \ldots, n, \overline{n}).$$

The following notations will also be used:

$$\begin{aligned}
NC_1(n) &= \{\sigma : \sigma \in NC(n), \sigma \text{ has only one exterior block}\} \\
NC_2(n) &= \{\sigma : \sigma \in NC(n), \sigma \text{ has only two exterior blocks}\} \\
NC_S(n) &= \{\sigma : \sigma \in NC(2n), \sigma \text{ the elements from the same block of } \sigma \text{ have} \\
&\quad \text{the same parity}\}.
\end{aligned}$$

For $\sigma \in NC_S(2n)$, denote σ_+, respectively σ_- the restriction of σ to the even, respectively odd, numbers from $\{1, 2, \ldots, 2n\}$. Define

$$NC_0(n) = \{\sigma : \sigma \in NC(n), \sigma_+ = Kr(\sigma_-)\}.$$

Also, we will need to consider the mappings

$$NC(n) \ni \sigma \mapsto \widehat{\sigma} \in NC(2n)$$

constructing by doubling the elements, and

$$NC(n) \times NC(m) \ni (\pi, \sigma) \mapsto \pi \oplus \sigma \in NC(m + n),$$

the juxtaposition of partitions.

3. THE ^cT-TRANSFORM

Let $\mathfrak{A}$ be a unital algebra, $\varphi, \psi : \mathfrak{A} \longrightarrow \mathbb{C}$ be two normalized linear functionals and X be n element from $\mathfrak{A}$. Denote by $m_X(z)$, respectively $M_X(z)$, the moment generating power series of X with respect to ψ and φ, as in the formulas (1), (2).

In this section we will construct a formal power series $^cT_X(z)$ such that:

(i) for $\varphi = \psi$ one has that $T_X(z) = {}^cT_X(z)$ (see 2.1 for the definition of $T_X(z)$).
(ii) $M_X(z)$ can be obtained from $T_X(z)$ and $^cT_X(z)$ via substitutional composition, substitutional inverse and algebraic operations.
(iii) if X and Y are c-free elements from $\mathfrak{A}$, then

$$^cT_{XY}(z) = {}^cT_X(z)^cT_Y(z).$$

The construction of $^cT_X(z)$ is presented as a natural consequence of the combinatorial properties of the free and c-free cumulants R^n and $^cR^n$.

For $a_1, \ldots, a_n \in \mathfrak{A}$ we need to consider the multilinear maps

$$\kappa_\pi[a_1, \ldots, a_n] = \prod_{\substack{B \in \pi \\ B = (i(1), \ldots, i(p))}} R^p(a_{i(1)} \cdots a_{i(p)})$$

$$\mathcal{K}_\pi[a_1, \ldots, a_n] = \left[\prod_{\substack{D \in Ext(\pi) \\ D = (i(1), \ldots, i(p))}} {}^cR^p(a_{i(1)} \cdots a_{i(p)})\right] \left[\prod_{\substack{B \in Int(\pi) \\ B = (i(1), \ldots, i(q))}} R^q(a_{i(1)} \cdots a_{i(q)})\right]$$

First, let us remark the following result on free and c-free cumulants with constants among entries:

Lemma 3.1. *For all $a_1, \ldots, a_n \in \mathfrak{A}$, one has that*

$$R^{n+1}(a_1, \ldots, a_j, 1, a_{j+1}, \ldots, a_n) = 0$$
$$^cR^{n+1}(a_1, \ldots, a_j, 1, a_{j+1}, \ldots, a_n) = 0$$

Proof. The first part is proved in [15]. We will prove the second part by induction. First, since $\varphi(a) = \varphi(a1)$, the definition of $^cR^n$ gives

$$^cR^2(a, 1) + {}^cR^1(a)1 = {}^cR^1(a)$$

therefore $^cR^2(a, 1) = 0$.

In general, $\varphi(a_1 \cdots a_n) = \varphi(a_1 \cdots a_j 1 a_{j+1} \cdots a_n)$. But

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \mathcal{K}_\pi[a_1, \ldots, a_n]$$

while

$$\varphi(a_1 \cdots a_j 1 a_{j+1} \cdots a_n) = \sum_{\pi \in NC(n+1)} \mathcal{K}_\pi[a_1, \ldots, a_j, 1, a_{j+1}, \ldots, a_n]$$

$$= \sum_{\substack{\pi \in NC(n+1) \\ (1) \in \pi}} \mathcal{K}_\pi[a_1, \ldots, a_j, 1, a_{j+1}, \ldots, a_n] + R^{n+1}(a_1, \ldots, 1, \ldots, a_n)$$

$$+ \sum_{\substack{\pi \in NC(n+1) \\ (1) \notin \pi \\ \pi \neq \mathbb{1}_{n+1}}} \mathcal{K}_\pi[a_1, \ldots, a_j, 1, a_{j+1}, \ldots, a_n]$$

The last term cancels from the induction hypothesis. Consider the bijection $\pi \mapsto \pi'$ between $\{\pi : \pi \in NC(n+1), (j) \in \pi\}$ and $NC(n)$, where π' is obtained by erasing the block (j) from π. It has the property that

$$\mathcal{K}_\pi[a_1, \ldots, a_j, 1, a_{j+1}, \ldots, a_n] = \mathcal{K}_{\pi'}[a_1, \ldots, a_n]$$

Therefore

$$\varphi(a_1 \cdots a_j 1 a_{j+1} \cdots a_n) = \varphi(a_1 \cdots a_n) + {}^cR^{n+1}[a_1, \ldots, a_j, 1, a_{j+1}, \ldots, a_n]$$

hence the conclusion. $\square$

We will focus next on free and c-free cumulants with products as entries.

Lemma 3.2. *Let $X, Y \in \mathfrak{A}$, c-free. For all $\pi \in NC(n)$ one has that:*

(a) $\kappa_\pi[XY, \ldots, XY] = \sum_{\substack{\sigma \in NC_S(2n) \\ \sigma \vee \widehat{0_n} = \widehat{\pi}}} \kappa_\sigma[X, Y, \ldots, X, Y]$.

Particularly,

$$R_{XY}^n = \sum_{\sigma \in NC_0(2n)} \kappa_\sigma[X, Y, \ldots, X, Y].$$

(b) $\mathcal{K}_\pi[XY, \ldots, XY] = \sum_{\substack{\sigma \in NC_S(2n) \\ \sigma \vee \widehat{0_n} = \widehat{\pi}}} \mathcal{K}_\sigma[X, Y, \ldots, X, Y]$.

Particularly,

$$^cR_{XY}^n = \sum_{\sigma \in NC_0(2n)} \mathcal{K}_\sigma[X, Y, \ldots, X, Y].$$

Proof. (a) is shown in [15]. The proof of (b) will be done by induction on n. For $n = 1$, the statement is trivial:

$$^{c}R^{1}[XY] = \varphi(XY) = \varphi(X \cdot Y) = {}^{c}R^{2}(X,Y) + {}^{c}R^{1}(X){}^{c}R^{1}(Y) = {}^{c}R^{1}(X){}^{c}R^{1}(Y)$$

sice the mixed c-free cumulants vanish (see 2.1).

For the induction step, we distinguish two cases:

Case 1: $\pi \neq \mathbb{1}_n$.

If π has more than one exterior block, then $\pi = \pi_1 \oplus \pi_2$ for some $m < n$, $\pi_1 \in NC(m), \pi_2 \in NC(n-m)$. One has that

$$\mathcal{K}_{\pi}[XY,\ldots,XY] = \mathcal{K}_{\pi_1}[XY,\ldots,XY]\mathcal{K}_{\pi_2}[XY,\ldots,XY],$$

and the result follows from the induction hypothesis.

If π has exactly one exterior block, then

$$\mathcal{K}_{\pi}[XY,\ldots,XY] = {}^{c}R^{p}[Xy,\ldots,XY]\kappa_{\overline{\pi}}[XY,\ldots,XY],$$

where p is the length of the exterior block of π and $\overline{\pi}$ is the non-crossing partition obtained by erasing the exterior block of π. The result follows from 3.2 and the induction hypothesis.

Case 2: $\pi = \mathbb{1}_n$.

We need to show that

$$(3) \qquad\qquad {}^{c}R^{n}_{XY} = \sum_{\sigma \in NC_0(2n)} \mathcal{K}_{\sigma}[X,Y,\ldots,X,Y].$$

Before proceeding with the proof, let us take a better look at the right-hand side of 3. Since any $\sigma \in NC_0(2n)$ has exactly 2 exterior blocks, one containing 1 and one containing $2n$, each $\mathcal{K}_{\sigma}[X,Y,\ldots,X,Y]$ will have exactly two factors of the type ${}^{c}R^{m}$, namely ${}^{c}R^{p}_{X}$ and ${}^{c}R^{q}_{Y}$, where p, q are, respectively, the length of the first and second exterior block of σ. Also, $\sigma \in NC_0(2n)$ implies that all other factors of $\mathcal{K}_{\sigma}[X,Y,\ldots,X,Y]$ are of the form R^{s}_{X} or R^{t}_{Y}.

One has that

$$\begin{aligned}
\varphi\left((XY)^n\right) &= \varphi(XY \cdot XY \cdots XY) \\
&= \sum_{\pi \in NC(n)} \mathcal{K}_{\pi}[XY,\ldots,XY] \\
&= {}^{c}R^{n}_{XY} + \sum_{\substack{\pi \in NC(n) \\ \pi \neq \mathbb{1}_n}} \mathcal{K}_{\pi}[XY,\ldots,XY]
\end{aligned}$$

On the other hand,

$$(4) \qquad \varphi\left((XY)^n\right) = \varphi(X \cdot Y \cdots X \cdot Y) = \sum_{\sigma \in NC(2n)} \mathcal{K}_{\sigma}[X,Y,\ldots,X,Y]$$

Since all the mixed cumulants vanish, 4 becomes:

$$\begin{aligned}
\varphi\left((XY)^n\right) &= \sum_{\sigma \in NC_S(2n)} \mathcal{K}_{\sigma}[X,Y,\ldots,X,Y] \\
&= \sum_{\sigma \in NC_0(2n)} \mathcal{K}_{\sigma}[X,Y,\ldots,X,Y] + \sum_{\substack{\sigma \in NC_S(2n) \\ \sigma \notin NC_0(2n)}} \mathcal{K}_{\sigma}[X,Y,\ldots,X,Y]
\end{aligned}$$

But

$NC_S(2n) = \bigcup_{\pi \in NC(n)} \{\sigma : \sigma \in NC_S(2n), \sigma \vee \widehat{0_n} = \widehat{\pi}\}$. Also, for $\sigma \in NC_s(2n)$, one has that $\sigma \in NC_0(2n)$ if and only if $\sigma \vee \widehat{0_n} = \mathbb{1}_{2n}$. Therefore:

$$NC_S(2n) \setminus NC_0(2n) = \bigcup_{\substack{\pi \in NC(n) \\ \pi \neq \mathbb{1}_n}} \{\sigma : \sigma \in NC_S(2n), \sigma \vee \widehat{0_n} = \widehat{\pi}\}.$$

It follows that

$$\sum_{\substack{\sigma \in NC_S(2n) \\ \sigma \notin NC_0(2n)}} \mathcal{K}_\sigma[X, Y, \ldots, X, Y] = \sum_{\substack{\pi \in NC(n) \\ \pi \neq \mathbb{1}_n}} \sum_{\substack{\sigma \in NC_S(2n) \\ \sigma \vee \widehat{0_n} = \widehat{\pi}}} \mathcal{K}_\sigma[X, Y, \ldots, X, Y]$$

$$= \sum_{\substack{\pi \in NC(n) \\ \pi \neq \mathbb{1}_n}} \mathcal{K}[XY, \ldots, XY].$$

so the proof is now complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

For stating the next result we need first a brief review of the operation ⊠ (boxed convolution), as defined in [15].

Consider the formal power series $f(z) = \sum_{n=1}^{\infty} \alpha_n z^n$. For $\pi \in NC(n)$, we define

$$Cf_\pi(f) = \prod_{B \in \pi} \alpha_{|B|}.$$

If $g(z) = \sum_{n=1}^{\infty} \beta_n z^n$ is another formal power series, we define their boxed convolution

$$f(z) \boxtimes g(z) = (f \boxtimes g)(z) = \sum_{n=1}^{\infty} \gamma_n z^n$$

by:

$$\gamma_n = \sum_{n \in NC(n)} Cf_\pi(f) \cdot Cf_{Kr(\pi)}(g).$$

We will need the following two results proved in [15]:

Lemma 3.3. *Suppose $\alpha_1 \neq 0$. Then*

$$f^{\langle -1 \rangle} \circ (f \boxtimes g) = \frac{1}{\alpha_1} \left(f \overset{\vee}{\boxtimes} g \right)$$

where

$$\left(f \overset{\vee}{\boxtimes} g \right)(z) = \sum_{n=1}^{\infty} \lambda_n z^n$$

for

$$\lambda_n = \sum_{\substack{\pi \in NC(n) \\ (1) \in \pi}} Cf_\pi(f) \cdot Cf_{Kr(\pi)}(g)$$

Lemma 3.4. *If X and Y are free, then*

$$R_{XY}(z) = R_X(z) \boxtimes R_Y(z).$$

Now we can proceed with the main theorems of this section. To ease the notations, say that $R_X(z) = \sum_{n=1}^{\infty} \alpha_n z^n$ and $R_Y(z) = \sum_{n=1}^{\infty} \beta_n z^n$.

Theorem 3.5. *If X, Y are c-free, then*

$$\frac{1}{z}{}^c R_{XY}(z) = \left[\left(\frac{1}{z}{}^c R_X\right) \circ \left(\frac{1}{\alpha_1} R_X \stackrel{\vee}{\boxplus} R_Y\right)\right]\left[\left(\frac{1}{z}{}^c R_Y\right) \circ \left(\frac{1}{\beta_1} R_Y \stackrel{\vee}{\boxplus} R_X\right)\right]$$

Proof. As shown before,

(5) $$\qquad\qquad {}^c R_{XY}^n = \sum_{\sigma \in NC_0(2n)} \mathcal{K}_\sigma[X, Y, \dots, X, Y].$$

Each $\sigma \in NC_0(2n)$ has exactly 2 exterior blocks, one consisting on $1 = b_1 < b_2 < \cdots < b_p$ and the other one consisting on $b_p + 1 = d_1 < d_2 < \cdots < d_q = 2n$. Let us denote π_k the restriction of σ to $(b_k + 1, b_k + 2, \dots, b_{k+1} - 1)$ and ω_l the restriction of σ to $(d_l + 1, d_l + 2, \dots, d_{l+1} - 1)$. Then:

$$\mathcal{K}_\sigma[X, Y, \dots, X, Y] = {}^c R_X^p \cdot \left(\prod_{k=1}^{p-1} \kappa_{\pi_k}[Y, X, \dots, X, Y]\right) \cdot {}^c R_Y^q \cdot \left(\prod_{l=1}^{q-1} \kappa_{\omega_l}[X, Y, \dots, X, Y, X]\right)$$

Sice $\sigma \in NC_0(2n)$, one has that $(1) \oplus \pi_k \in NC_0(b_{k+1} - b_k + 1)$ and $((1) \oplus \omega_l \in NC_0(d_{l+1} - d_l + 1)$, therefore:

$$\kappa_{\pi_k}[Y, X, \dots, Y] \;=\; \frac{1}{\alpha_1}\kappa_{(1) \oplus \pi_k}[X, Y, \dots, X, Y]$$
$$\;=\; \frac{1}{\alpha_1}Cf_{((1)\oplus\pi_k)_-}(R_X) \cdot Cf_{Kr(((1)\oplus\pi_k)_-)}(R_Y)$$

and

$$\kappa_{\omega_l}[X, Y, X, \dots, X] \;=\; \frac{1}{\beta_1}\kappa_{(1) \oplus \omega_l}[X, Y, \dots, X, Y]$$
$$\;=\; \frac{1}{\beta_1}Cf_{((1)\oplus\omega_l)_-}(R_Y) \cdot Cf_{Kr(((1)\oplus\omega_l)_-)}(R_X)$$

hence q.e.d.. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 3.6. *If X, Y are c-free such that $psi(X) \neq 0 \neq \psi(Y)$, then*

$$\left[\left(\frac{1}{z}{}^c R_{XY}\right) \circ \left(R_{XY}^{\langle -1\rangle}(z)\right)\right] = \left[\left(\frac{1}{z}{}^c R_X\right) \circ \left(R_X^{\langle -1\rangle}(z)\right)\right] \cdot \left[\left(\frac{1}{z}{}^c R_Y\right) \circ \left(R_Y^{\langle -1\rangle}(z)\right)\right].$$

Proof. From 3.5, one has:

$$\frac{1}{z}{}^c R_{XY}(z) = \left[\left(\frac{1}{z}{}^c R_X\right) \circ \left(\frac{1}{\alpha_1} R_X \stackrel{\vee}{\boxplus} R_Y\right)\right]\left[\left(\frac{1}{z}{}^c R_Y\right) \circ \left(\frac{1}{\beta_1} R_Y \stackrel{\vee}{\boxplus} R_X\right)\right]$$

Using 3.3, the above equality becomes

$$\frac{1}{z}{}^c R_{XY}(z) \;=\; \left[\left(\frac{1}{z}{}^c R_X\right) \circ \left(R_X^{\langle -1\rangle} \circ \left(R_X \stackrel{\vee}{\boxplus} R_Y\right)\right)\right]\left[\left(\frac{1}{z}{}^c R_Y\right) \circ \left(R_Y^{\langle -1\rangle} \circ \left(R_Y \stackrel{\vee}{\boxplus} R_X\right)\right)\right]$$
$$\;=\; \left[\left(\left(\frac{1}{z}{}^c R_X\right) \circ R_X^{\langle -1\rangle}\right) \circ \left(R_X \stackrel{\vee}{\boxplus} R_Y\right)\right]\left[\left(\left(\frac{1}{z}{}^c R_Y\right) \circ R_Y^{\langle -1\rangle}\right) \circ \left(R_Y \stackrel{\vee}{\boxplus} R_X\right)\right]$$

But $R_{XY} = r_X \boxtimes R_y$, so

$$\frac{1}{z}{}^c R_{XY}(z) = \left[\left(\left(\frac{1}{z}{}^c R_X\right) \circ R_X^{\langle -1\rangle}\right) \circ (R_{XY})\right]\left[\left(\left(\frac{1}{z}{}^c R_Y\right) \circ R_Y^{\langle -1\rangle}\right) \circ (R_{XY})\right]$$

Finally, composing at the right to $R_{XY}^{\langle -1\rangle}$ one gets the conclusion. $\qquad\square$

Corollary 3.7. *The power series ${}^c T_x(z)$ defined by*

$$ {}^c T_X(z) = \left(\frac{1}{z}{}^c R_X(z)\right) \circ \left(R^{\langle -1\rangle}(z)\right) $$

satisfies the properties (i)-(iii) described in the beginning of the section.

Proof. (iii) is proved in Theorem 3.6. (i) is an immediate consequence of comparing the definitions of T_X (as in Lemma 2.1) and of ${}^c T_X$. Finally, (ii) is implied by (1), (2) and the definition of ${}^c T_X$. $\qquad\square$

4. Multiplicative conditionally free convolution of measures

Denote by $\mathcal{M}_{\mathbb{T}}$ the family of all Borel probability measures supported on the unit circle $\mathbb{T}$ and by $\mathcal{M}_{\mathbb{T}}^\times$ the set of all measures $\nu \in \mathcal{M}_{\mathbb{T}}$ such that $\int_{\mathbb{T}} \zeta\, d\nu(\zeta) \neq 0$.

For $\mu \in \mathcal{M}_{\mathbb{T}}$ we denote

$$m_\mu(z) = \int_{\mathbb{T}} \frac{z\zeta}{1 - z\zeta} d\mu(\zeta)$$

the moment generating function of μ, analytic in the unit disk $\mathbb{D}$.

Consider now $\mu \in \mathcal{M}_{\mathbb{T}}$ and $\nu \in \mathcal{M}_{\mathbb{T}}^\times$. To the pair (μ, ν) we associate the functions $R_\nu(z)$, ${}^c R_{(\mu,\nu)}(z)$, $T_\nu(z)$, ${}^c T_{(\mu,\nu)}(z)$, analytic in some neighborhood of zero, and given by the following relations:

$$
\begin{aligned}
R_\nu\left(z[1 + m_\nu(z)]\right) &= m_\nu(z) \\
{}^c R_{(\mu,\nu)}\left(z[1 + m_\nu(z)) \cdot (1 + m_\mu(z))\right) &= m_\mu(z) \cdot (1 + m_\nu(z)) \\
T_\nu(z) &= \left(\frac{1}{z} R_\nu(z)\right) \circ \left(R_\nu^{\langle -1\rangle}(z)\right) \\
{}^c T_{(\mu,\nu)}(z) &= \left(\frac{1}{z}{}^c R_{\mu,\nu)}(z)\right) \circ \left(R_\nu^{\langle -1\rangle}(z)\right)
\end{aligned}
$$

Definition 4.1. *If $(\mu_1, \nu_1), (\mu_2, \nu_2) \in \mathcal{M}_{\mathbb{T}} \times \mathcal{M}_{\mathbb{T}}^\times$, their multiplicative conditionally free convolution, $(\mu_1, \nu_1) \boxtimes (\mu_2, \nu_2)$ is the unique pair $(\mu, \nu) \in \mathcal{M}_{\mathbb{T}} \times \mathcal{M}_{\mathbb{T}}^\times$ such that $T_\nu(z) = T_{\nu_1}(z) \cdot T_{\nu_2}(z)$ and ${}^c T_{(\mu,\nu)}(z) = {}^c T_{(\mu_1,\nu_1)}(z) \cdot {}^c T_{(\mu_2,\nu_2)}(z)$.*

Particularly, for X, Y c-free elements of $\mathfrak{A}$, if $(\mu_1, \nu_1), (\mu_2, \nu_2)$ are the distributions of X, respectively Y, then $(\mu_1, \nu_1) \boxtimes (\mu_2, \nu_2)$ is the distribution of XY.

The next two sections address some analytical properties of the operation $\boxtimes$. More precisely, Section 5 states some limit theorems and Section 6 describes $\boxtimes$ infinite divisibility.

It will be convenient to introduce a variation of the function ${}^c T_{(\mu,\nu)}$ as follows. Namely, for the measures $\mu \in \mathcal{M}_{\mathbb{T}}$ and $\nu \in \mathcal{M}_{\mathbb{T}}^\times$, we consider the analytic function

$$(6) \qquad \Sigma_{(\mu,\nu)}(z) = {}^c T_{(\mu,\nu)}\left(\frac{z}{1 - z}\right)$$

in a neighborhood of zero.

Similarly to Definition 2.2, we denote $\eta_\mu(z) = \dfrac{m_\mu(z)}{1 + m_\mu(z)}$ and $B_\mu(z) = \dfrac{m_\mu(z)}{z(1 + m_\mu(z))}$

Lemma 4.2. *For $\eta_\nu(z)$ and $B_\mu(z)$, defined above, one has:*

$$\Sigma_{(\mu,\nu)} = \left[\frac{1}{z}\eta_\mu(z)\right] \circ \left[\eta_\nu^{\langle-1\rangle}(z)\right] = B_\mu\left(\eta_\nu^{\langle-1\rangle}(z)\right)$$

Proof. The relation $R_\nu\left(z[1 + m_\nu(z)]\right) = m_\nu(z)$ (i. e. the definition of R_ν) implies that

$$R_\nu^{\langle-1\rangle}(z) = (1+z)m_\nu^{\langle-1\rangle}(z),$$

hence

(7) $$R_\nu^{\langle-1\rangle}(m_\nu(z)) = z(1 + m_\nu(z))$$

Let us now consider the definition of ${}^cR_{(\mu,\nu)}$,

$${}^cR_{(\mu,\nu)}\left(z[1 + m_\nu(z)) \cdot (1 + m_\mu(z)\right) = m_\mu(z) \cdot (1 + m_\nu(z))$$

which implies that

$$\frac{{}^cR_{(\mu,\nu)}\left(z(1 + m_n u(z))\right)}{z(1 + m_n u(z))} = \frac{m_m u(z)}{z(1 + m_m u(z))}.$$

Taking into consideration (7) it follows that

$$\frac{{}^cR_{(\mu,\nu)}\left(R_\nu^{\langle-1\rangle}(m_\nu(z))\right)}{R_\nu^{\langle-1\rangle}(m_\nu(z))} = B_\mu(z),$$

that is

$${}^cT_{(\mu,\nu)}(m_\nu(z)) = B_\mu(z)$$

and composing at right with $m_\nu^{\langle-1\rangle}$ we get the conclusion. $\qquad\square$

Note that

$$\Sigma_{(\mu,\nu)}(0) = B_\mu(0) = \int_{\mathbb{T}} \zeta \, d\mu(\zeta)$$

and the function $\Sigma_{(\delta_\lambda,\nu)}(z)$ is the constant function λ.

As observed in [1], the function B_μ maps $\mathbb{D}$ into $\overline{\mathbb{D}}$, and, conversely, any analytic function $B : \mathbb{D} \to \overline{\mathbb{D}}$ is of the form B_μ for a unique probability measure μ on $\mathbb{T}$. As a consequence of this observation, the function $\Sigma_{(\mu,\nu)}$ is uniformly bounded by 1. Moreover, if $\nu \in \mathcal{M}_{\mathbb{T}}^\times$ and Σ is an analytic function defined on the set $\eta_\nu(\mathbb{D})$ such that the function Σ is uniformly bounded by 1, then there exists a unique probability measure μ on $\mathbb{T}$ such that $\Sigma = \Sigma_{(\mu,\nu)}$.

Before starting Section 5, we will finally mention the following construction. If μ, ν are two probability measures on $\mathbb{T}$, their *boolean convolution* $\mu \uplus \nu$ is the unique measure on $\mathbb{T}$ given by

$$B_{\mu \uplus \nu}(z) = B_\mu(z)B_\nu(z)$$

5. Limit Theorems

Let $\{k_n\}_{n=1}^\infty$ be a sequence of positive integers. Consider two *infinitesimal* triangular arrays $\{\mu_{nk} : n \in \mathbb{N}, 1 \le k \le k_n\}$ and $\{\nu_{nk} : n \in \mathbb{N}, 1 \le k \le k_n\}$ in $\mathcal{M}_\mathbb{T}^\times$. Here the infinitesimality means that

$$\lim_{n\to\infty} \max_{1 \le k \le k_n} \mu_{nk}(\{\zeta \in \mathbb{T} : |\zeta - 1| \ge \varepsilon\}) = 0,$$

for every $\varepsilon > 0$. We say that a sequence $\{\mu_n\}_{n=1}^\infty \subset \mathcal{M}_\mathbb{T}$ converges *weakly* to a measure $\mu \in \mathcal{M}_\mathbb{T}$ if

$$\lim_{n\to\infty} \int_\mathbb{T} f(\zeta)\, d\mu_n(\zeta) = \int_\mathbb{T} f(\zeta)\, d\mu(\zeta),$$

for every continuous function f on $\mathbb{T}$. The weak convergence of a sequence of pairs $\{(\mu_n, \nu_n)\}_{n=1}^\infty$ simply means the componentwise weak convergence. Let $\{\lambda_n\}_{n=1}^\infty$ and $\{\lambda'_n\}_{n=1}^\infty$ be two sequences in $\mathbb{T}$. The aim of current section is to study the limiting behavior of the sequence of pairs $\{(\mu_n, \nu_n)\}_{n=1}^\infty$, where

$$(\mu_n, \nu_n) = (\delta_{\lambda_n}, \delta_{\lambda'_n}) \boxtimes (\mu_{n1}, \nu_{n1}) \boxtimes (\mu_{n2}, \nu_{n2}) \boxtimes \cdots \boxtimes (\mu_{nk_n}, \nu_{nk_n}),$$

for δ_λ the Dirac point mass supported at $\lambda \in \mathbb{T}$.

We would like to mention at this point that the asymptotic behavior of boolean convolution $\uplus$ and that of free convolution $\boxtimes$ has been studied thoroughly in [20], where the necessary and sufficient conditions for the weak convergence were found. These conditions show that the sequence $\delta_{\lambda_n} \uplus \mu_{n1} \uplus \mu_{n2} \uplus \cdots \uplus \mu_{nk_n}$ converges weakly if and only if the sequence $\delta_{\lambda_n} \boxtimes \mu_{n1} \boxtimes \mu_{n2} \boxtimes \cdots \boxtimes \mu_{nk_n}$ does, provided that the array $\{\mu_{nk}\}_{n,k}$ is infinitesimal. Moreover, the limit laws are proved to be infinitely divisible (see [20]), and the boolean and free limits are related in a quite explicit manner. In the sequel, we will prove the analogous results for c-free and boolean convolutions.

To our purposes, we also mention the characterization of infinite divisibility relative to boolean convolution $\uplus$ and that relative to free convolution $\boxtimes$. A measure $\nu \in \mathcal{M}_\mathbb{T}$ is $\uplus$-*infinitely divisible* if, for each $n \in \mathbb{N}$, there exists $\nu_n \in \mathcal{M}_\mathbb{T}$ such that

$$\nu = \underbrace{\nu_n \uplus \nu_n \uplus \cdots \uplus \nu_n}_{n \text{ times}}.$$

The notion of $\boxtimes$-infinite divisibility for a measure is defined analogously.

As shown in [11], a measure $\nu \in \mathcal{M}_\mathbb{T}$ is $\uplus$-infinitely divisible if and only if either ν is the Haar measure on $\mathbb{T}$, or the function B_ν can be expressed as

$$B_\nu(z) = \gamma \exp\left(-\int_\mathbb{T} \frac{1 + \zeta z}{1 - \zeta z}\, d\sigma(\zeta)\right), \qquad z \in \mathbb{D},$$

where $\gamma \in \mathbb{T}$ and σ is a finite positive Borel measure on $\mathbb{T}$. In other words, ν is $\uplus$-infinitely divisible if and only if either the function $B_\nu(z) = 0$ for all z in $\mathbb{D}$, or $0 \notin B_\nu(\mathbb{D})$. The notation $\nu_\uplus^{\gamma,\sigma}$ will denote the $\uplus$-infinitely divisible measure determined by γ and σ via the above formula.

Analogously, a measure $\nu \in \mathcal{M}_\mathbb{T}^\times$ is $\boxtimes$-infinitely divisible if and only if the function $\eta_\nu^{\langle -1 \rangle}$ can be written as

$$\eta_\nu^{\langle -1 \rangle}(z) = z \cdot \gamma \exp\left(\int_\mathbb{T} \frac{1 + \zeta z}{1 - \zeta z}\, d\sigma(\zeta)\right), \qquad z \in \mathbb{D},$$

for some $\gamma \in \mathbb{T}$ and a finite positive Borel measure σ on $\mathbb{T}$. The $\boxtimes$-infinitely divisible measure ν described above will be denoted by $\nu_{\boxtimes}^{\gamma, \sigma}$. The Haar measure m is the only $\boxtimes$-infinitely divisible measure on $\mathbb{T}$ with zero first moment.

Let us proceed to the proof of the limit theorems for c-free convolution. We first show that weak convergence of probability measures can be translated into convergence properties of corresponding functions B and Σ.

Proposition 5.1. *Let $\{\mu_n\}_{n=1}^{\infty} \subset \mathcal{M}_{\mathbb{T}}$ and $\{\nu_n\}_{n=1}^{\infty} \subset \mathcal{M}_{\mathbb{T}}^{\times}$ be two sequences of probability measures, and let μ be a probability measure supported on $\mathbb{T}$.*

   (i) *The sequence μ_n converges weakly to μ if and only if the functions B_{μ_n} converge uniformly on compact subsets of $\mathbb{D}$ to the function B_μ.*
   (ii) *Suppose that $\{\nu_n\}_{n=1}^{\infty}$ converges weakly to a measure $\nu \in \mathcal{M}_{\mathbb{T}}^{\times}$. Then the sequence μ_n converges weakly to μ if and only if there exists a neighborhood of zero $\mathcal{D} \subset \mathbb{D}$ such that all functions $\Sigma_{(\mu_n, \nu_n)}$ and $\Sigma_{(\mu, \nu)}$ are defined in $\mathcal{D}$, and the functions $\Sigma_{(\mu_n, \nu_n)}$ converge uniformly on $\mathcal{D}$ to the function $\Sigma_{(\mu, \nu)}$.*

Proof. The equivalence in (i) has been observed in [20], and it is based on the following identity:

$$(8) \qquad \frac{1 + z B_\mu(z)}{1 - z B_\mu(z)} = \int_{\mathbb{T}} \frac{1 + \zeta z}{1 - \zeta z} \, d\mu(\zeta), \qquad z \in \mathbb{D},$$

which says that the Poisson integral of the measure $d\mu(\overline{\zeta})$ is determined by the function B_μ.

Let us prove now (ii). Assume that $\{\mu_n\}_{n=1}^{\infty}$ converges weakly to μ. Then the hypothesis on weak convergence of $\{\nu_n\}_{n=1}^{\infty}$ implies that there exists a neighborhood of zero $\mathcal{D} \subset \mathbb{D}$ such that the functions $\eta_{\nu_n}^{\langle -1 \rangle}$ and $\eta_{\nu}^{\langle -1 \rangle}$ are defined in $\mathcal{D}$ (hence so are the functions $\Sigma_{(\mu_n, \nu_n)}$ and $\Sigma_{(\mu, \nu)}$), and the sequence $\eta_{\nu_n}^{\langle -1 \rangle}$ converges uniformly on $\mathcal{D}$ to $\eta_{\nu}^{\langle -1 \rangle}$. It follows that there exist $N \in \mathbb{N}$ and a disk $\mathcal{D}' \subset \mathbb{D}$ such that $\eta_{\nu}^{\langle -1 \rangle}(\mathcal{D}) \subset \mathcal{D}'$ and $\eta_{\nu_n}^{\langle -1 \rangle}(\mathcal{D}) \subset \mathcal{D}'$ for every $n > N$. Also, the Cauchy estimate implies that there exists $K = K(\mathcal{D}') > 0$ so that the derivatives $\left| B'_{\mu_n}(z) \right| \leq K$ for every $n \in \mathbb{N}$ and $z \in \mathcal{D}'$. Therefore, we have

$$
\begin{aligned}
\left| \Sigma_{(\mu_n, \nu_n)}(z) - \Sigma_{(\mu, \nu)}(z) \right| &\leq \left| B_{\mu_n}\left(\eta_{\nu_n}^{\langle -1 \rangle}(z) \right) - B_{\mu_n}\left(\eta_{\nu}^{\langle -1 \rangle}(z) \right) \right| \\
&\quad + \left| B_{\mu_n}\left(\eta_{\nu}^{\langle -1 \rangle}(z) \right) - B_{\mu}\left(\eta_{\nu}^{\langle -1 \rangle}(z) \right) \right| \\
&\leq K \left| \eta_{\nu_n}^{\langle -1 \rangle}(z) - \eta_{\nu}^{\langle -1 \rangle}(z) \right| + \left| B_{\mu_n}\left(\eta_{\nu}^{\langle -1 \rangle}(z) \right) - B_{\mu}\left(\eta_{\nu}^{\langle -1 \rangle}(z) \right) \right|,
\end{aligned}
$$

for every $n > N$ and $z \in \mathcal{D}$. Then (i) implies that the functions $\Sigma_{(\mu_n, \nu_n)}$ converge uniformly on $\mathcal{D}$ to the function $\Sigma_{(\mu, \nu)}$.

Conversely, suppose that the functions $\Sigma_{(\mu_n, \nu_n)}$ converge uniformly on $\mathcal{D}$ to the function $\Sigma_{(\mu, \nu)}$. Observe that $B_{\mu_n}(z) = \Sigma_{(\mu_n, \nu_n)}(\eta_{\nu_n}(z))$ for all $z \in \mathbb{D}$. A similar argument as in the previous paragraph shows that there exists a positive constant K' such that the estimate

$$|B_{\mu_n}(z) - B_\mu(z)| \leq K' \left| \eta_{\nu_n}(z) - \eta_\nu(z) \right| + \left| \Sigma_{(\mu_n, \nu_n)}\left(\eta_\nu(z) \right) - \Sigma_{(\mu, \nu)}\left(\eta_\nu(z) \right) \right|$$

holds in a neighborhood of zero, for sufficiently large n. Therefore, the weak convergence of $\{\nu_n\}_{n=1}^{\infty}$ implies that the sequence $B_{\mu_n}(z)$ converges uniformly on a neighborhood of zero to the function $B_\mu(z)$. Moreover, this convergence is actually

uniform on any compact subset of $\mathbb{D}$ by an easy application of Montel's theorem. We therefore conclude, from (i), that the sequence μ_n converges weakly to μ. $\square$

For an infinitesimal array $\{\mu_{nk}\}_{n,k}$, we define the complex numbers $b_{nk} \in \mathbb{T}$ by

$$(9) \qquad b_{nk} = \exp\left(i \int_{|\arg \zeta| < 1} \arg \zeta \, d\mu_{nk}(\zeta)\right)$$

where $\arg \zeta$ denotes the principal value of the argument of ζ, and the probability measures μ_{nk}° by

$$d\mu_{nk}^\circ(\zeta) = d\mu_{nk}(b_{nk}\zeta).$$

The array $\{\mu_{nk}^\circ\}_{n,k}$ is infinitesimal and $\lim_{n \to \infty} \max_{1 \le k \le k_n} |\arg b_{nk}| = 0$. We also associate to each measure μ_{nk}° an analytic function

$$h_{nk}(z) = -i \int_{\mathbb{T}} \Im\zeta \, d\mu_{nk}^\circ(\zeta) + \int_{\mathbb{T}} \frac{1+\zeta z}{1-\zeta z}(1 - \Re\zeta) \, d\mu_{nk}^\circ(\zeta), \qquad z \in \mathbb{D},$$

and observe that $\Re h_{nk}(z) > 0$ for all $z \in \mathbb{D}$ unless the measure $\mu_{nk}^\circ = \delta_1$.

Proposition 5.2. *Suppose that $\mathcal{D} \subset \mathbb{D}$ is a disk centered at zero with radius less than $1/4$. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence in $\mathbb{T}$, and let $\{\mu_{nk}\}_{n,k}$ be an infinitesimal array in $\mathcal{M}_{\mathbb{T}}^\times$. Then we have:*

(1) *$1 - B_{\mu_{nk}^\circ}(z) = h_{nk}\left(\overline{b_{nk}}z\right)(1 + o(1))$ uniformly in k and $z \in \mathcal{D}$ as $n \to \infty$.*

(2) *There exists a constant $L = L(\mathcal{D}) > 0$ such that for every n and k we have*

$$|h_{nk}(w) - h_{nk}(z)| \le L\,|h_{nk}(z)|\,|w - z|, \qquad z, w \in \mathcal{D}.$$

(3) *The sequence of functions $\{\exp(i \arg \lambda_n + i \sum_{k=1}^{k_n} \arg b_{nk} - \sum_{k=1}^{k_n} h_{nk}(z))\}_{n=1}^\infty$ converges uniformly on compact subsets of $\mathbb{D}$ if and only if the sequence of functions $\{\lambda_n \prod_{k=1}^{k_n} B_{\mu_{nk}}(z)\}_{n=1}^\infty$ does. Moreover, the two sequences have the same limit function.*

Proof. (1) and (3) are proved in [20]. To prove (2), let us consider the analytic function

$$h_\mu(z) = \int_{\mathbb{T}} \frac{1+\zeta z}{1-\zeta z}(1 - \Re\zeta) d\mu(\zeta), \qquad z \in \mathbb{D},$$

for a measure $\mu \in \mathcal{M}_{\mathbb{T}}$. For $z, w \in \mathcal{D}$, we have

$$
\begin{aligned}
|h_\mu(w) - h_\mu(z)| &\le |w - z| \int_{\mathbb{T}} \frac{2}{(1-|z|)(1-|w|)}(1 - \Re\zeta) d\mu(\zeta) \\
&\le 4\,|w - z| \int_{\mathbb{T}}(1 - \Re\zeta) d\mu(\zeta).
\end{aligned}
$$

In addition, Harnack's inequality shows that there exists $M = M(\mathcal{D}) > 0$ such that

$$\Re\left[\frac{1+\zeta z}{1-\zeta z}\right] \ge M, \qquad z \in \mathcal{D},\ \zeta \in \mathbb{T}.$$

Therefore, we deduce that

$$M \int_{\mathbb{T}}(1 - \Re\zeta) d\mu(\zeta) \le \int_{\mathbb{T}} \Re\left[\frac{1+\zeta z}{1-\zeta z}\right](1 - \Re\zeta) d\mu(\zeta) = \Re h_\mu(z) \le |h_\mu(z)|,$$

for every $z \in \mathcal{D}$. (2) follows from these considerations. $\square$

Lemma 5.3. *For sufficiently large n, there exists a disk $\mathcal{D} \subset \mathbb{D}$ centered at zero such that*

$$1 - \overline{b_{nk}}\Sigma_{(\mu_{nk},\nu_{nk})}(z) = h_{nk}(z)(1 + u_{nk}(z)), \qquad z \in \mathcal{D},\, 1 \leq k \leq k_n,$$

where the limit

$$\lim_{n\to\infty} \max_{1\leq k\leq k_n} |u_{nk}(z)| = 0$$

holds uniformly in $\mathcal{D}$.

Proof. Introduce measures

$$d\nu_{nk}^{\circ}(\zeta) = d\nu_{nk}(b_{nk}\zeta),$$

where the complex numbers b_{nk} are defined as in (10). It was shown in [2] that the limits $\lim_{n\to\infty} \eta_{\nu_{nk}}(z) = z$ and $\lim_{n\to\infty} \eta_{\nu_{nk}^{\circ}}(z) = z$ hold uniformly in k and on compact subsets of $\mathbb{D}$. In particular, it follows that, as n tends to infinity, there exists a disk $\mathcal{D} \subset \mathbb{D}$ centered at zero such that the functions $\Sigma_{(\mu_{nk},\nu_{nk})}$ and $\Sigma_{(\mu_{nk}^{\circ},\nu_{nk}^{\circ})}$ are both defined in $\mathcal{D}$ and $\eta_{\nu_{nk}^{\circ}}^{\langle-1\rangle}(z) = z(1 + o(1))$ uniformly in k and $z \in \mathcal{D}$.

Then the desired result follows form 5.2(i), 5.2(ii), and the following observation:

$$\Sigma_{(\mu_{nk},\nu_{nk})}(z) = b_{nk}\Sigma_{(\mu_{nk}^{\circ},\nu_{nk}^{\circ})}(z) = b_{nk}B_{\mu_{nk}^{\circ}}\left(\eta_{\nu_{nk}^{\circ}}^{\langle-1\rangle}(z)\right) = b_{nk}B_{\mu_{nk}^{\circ}}\left(b_{nk}\eta_{\nu_{nk}}^{\langle-1\rangle}(z)\right).$$

$\square$

As shown in [5], for every neighborhood of zero $\mathcal{D} \subset \mathbb{D}$ there exists $M = M(\mathcal{D}) > 0$ such that

$$|\Im h_{nk}(z)| \leq M\Re h_{nk}(z), \qquad z \in \mathcal{D},\, 1 \leq k \leq k_n,$$

for sufficiently large n.

Suppose that $\mathcal{D} \subset \mathbb{D}$ is a neighborhood of zero. The infinitesimality of the arrays $\{\mu_{nk}\}_{n,k}$ and $\{\nu_{nk}\}_{n,k}$ implies that the functions $\Sigma_{(\mu_{nk},\nu_{nk})}(z)$ converge uniformly in k and $z \in \mathcal{D}$ to 1 as $n \to \infty$. It follows that the principal branch of $\log \Sigma_{(\mu_{nk},\nu_{nk})}(z)$ is defined in $\mathcal{D}$ for large n. Moreover, we have

$$\log \overline{b_{nk}}\Sigma_{(\mu_{nk},\nu_{nk})}(z) = \left[\overline{b_{nk}}\Sigma_{(\mu_{nk},\nu_{nk})}(z) - 1\right](1 + o(1))$$

uniformly in k and $z \in \mathcal{D}$ when n is sufficiently large.

To prove Theorem 5.5, we will need again an auxiliary result from [20]:

Lemma 5.4. *Consider a sequence $\{r_n\}_{n=1}^{\infty} \subset \mathbb{R}$ and two triangular arrays $\{z_{nk}\}_{n,k}$ and $\{w_{nk}\}_{n,k}$ of complex numbers. Suppose that*

(1) *$\Re w_{nk} \leq 0$ and $\Re z_{nk} \leq 0$ for all n and k;*
(2) *$z_{nk} = w_{nk}(1 + \varepsilon_{nk})$, where $\lim_{n\to\infty} \max_{1\leq k\leq k_n} |\varepsilon_{nk}| = 0$;*
(3) *there exists a constant $M > 0$ such that $|\Im w_{nk}| \leq M |\Re w_{nk}|$ for all n, k.*

Then the sequence $\{\exp(ir_n + \sum_{k=1}^{k_n} z_{nk})\}_{n=1}^{\infty}$ converges if and only if the sequence $\{\exp(ir_n + \sum_{k=1}^{k_n} w_{nk})\}_{n=1}^{\infty}$ does. Moreover, the two sequences have the same limit.

Theorem 5.5. *Let $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\lambda_n'\}_{n=1}^{\infty}$ be two sequences in $\mathbb{T}$, and let $\{\mu_{nk}\}_{n,k}$ and $\{\nu_{nk}\}_{n,k}$ be two infinitesimal arrays in $\mathcal{M}_{\mathbb{T}}^{\times}$. Assume that $\mathcal{D} \subset \mathbb{D}$ is a neighborhood of zero. Then the sequence of functions $\{\lambda_n \prod_{k=1}^{k_n} \Sigma_{(\mu_{nk},\nu_{nk})}(z)\}_{n=1}^{\infty}$ converges uniformly on $\mathcal{D}$ if and only if the sequence of functions $\{\lambda_n \prod_{k=1}^{k_n} B_{\mu_{nk}}(z)\}_{n=1}^{\infty}$ does. Moreover, the two sequences have the same limit function.*

Proof. For sufficiently large n and $z \in \mathcal{D}$, let us write

$$\lambda_n \prod_{k=1}^{k_n} \Sigma_{(\mu_{nk}, \nu_{nk})}(z) = \exp\left(i \arg \lambda_n + i \sum_{k=1}^{k_n} \arg b_{nk} + \sum_{k=1}^{k_n} \log \overline{b_{nk}} \Sigma_{(\mu_{nk}, \nu_{nk})}(z)\right).$$

Applying Lemmas 5.3 and 5.4 to the arrays $\{\log \overline{b_{nk}} \Sigma_{(\mu_{nk}, \nu_{nk})}(z)\}_{n,k}$ and $\{-h_{nk}(z)\}_{n,k}$, we conclude that the sequences $\{\exp(i \arg \lambda_n + i \sum_{k=1}^{k_n} \arg b_{nk} - \sum_{k=1}^{k_n} h_{nk}(z))\}_{n=1}^{\infty}$ and $\{\lambda_n \prod_{k=1}^{k_n} \Sigma_{(\mu_{nk}, \nu_{nk})}(z)\}_{n=1}^{\infty}$ have the same asymptotic behavior as $n \to \infty$. The desired result follows from the fact that the above sequences are normal families of analytic functions, and from 5.2(3). $\qquad\square$

Fix now $\gamma, \gamma' \in \mathbb{T}$ and finite positive Borel measures σ, σ' on $\mathbb{T}$. Recall that $\nu_{\uplus}^{\gamma, \sigma}$ (resp., $\nu_{\boxtimes}^{\gamma', \sigma'}$) is the $\uplus$ (resp., $\boxtimes$)-infinitely divisible measure we have seen earlier.

Theorem 5.6. *Let $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\lambda_n'\}_{n=1}^{\infty}$ be two sequences in $\mathbb{T}$, and let $\{\mu_{nk}\}_{n,k}$ and $\{\nu_{nk}\}_{n,k}$ be two infinitesimal arrays in $\mathcal{M}_{\mathbb{T}}^{\times}$. Define*

$$(\mu_n, \nu_n) = (\delta_{\lambda_n}, \delta_{\lambda_n'}) \boxtimes (\mu_{n1}, \nu_{n1}) \boxtimes (\mu_{n2}, \nu_{n2}) \boxtimes \cdots \boxtimes (\mu_{nk_n}, \nu_{nk_n})$$

and $\rho_n = \delta_{\lambda_n} \uplus \mu_{n1} \uplus \mu_{n2} \uplus \cdots \uplus \mu_{nk_n}$, for every $n \in \mathbb{N}$. Suppose that $\{\nu_n\}_{n=1}^{\infty}$ converges weakly to $\nu_{\boxtimes}^{\gamma', \sigma'}$. Then the following statements are equivalent:

(1) *The sequence μ_n converges weakly to a measure $\mu \in \mathcal{M}_{\mathbb{T}}^{\times}$.*
(2) *The sequence ρ_n converges weakly to $\nu_{\uplus}^{\gamma, \sigma}$.*
(3) *The sequence of measures*

$$d\sigma_n(\zeta) = \sum_{k=1}^{k_n} (1 - \Re\zeta) \, d\mu_{nk}^{\circ}(\zeta)$$

converges weakly on $\mathbb{T}$ to the measure σ, and the sequence of complex numbers

$$\gamma_n = \exp\left(i \arg \lambda_n + i \sum_{k=1}^{k_n} \arg b_{nk} + i \sum_{k=1}^{k_n} \int_{\mathbb{T}} \Im\zeta \, d\mu_{nk}^{\circ}(\zeta)\right)$$

converges to γ as $n \to \infty$.

Moreover, if (1)-(3) are satisfied, then we have

$$\Sigma_{\left(\mu, \nu_{\boxtimes}^{\gamma', \sigma'}\right)}(z) = B_{\nu_{\uplus}^{\gamma, \sigma}}(z)$$

in a neighborhood of zero.

Proof. The equivalence between (2) and (3) was proved in [20]. We will show the equivalence of (1) and (2). Note first that we have

$$\Sigma_{(\mu_n, \nu_n)}(z) = \lambda_n \prod_{k=1}^{k_n} \Sigma_{(\mu_{nk}, \nu_{nk})}(z)$$

in a neighborhood of zero $\mathcal{D} \subset \mathbb{D}$ and

$$B_{\rho_n}(z) = \lambda_n \prod_{k=1}^{k_n} B_{\mu_{nk}}(z), \qquad z \in \mathbb{D}.$$

Suppose that (1) holds. By Proposition 5.1, we have

$$\lim_{n \to \infty} \Sigma_{(\mu_n, \nu_n)}(z) = \Sigma_{\left(\mu, \nu_{\boxtimes}^{\gamma', \sigma'}\right)}(z)$$

uniformly on $\mathcal{D}$. Then Theorem 5.5 implies that

$$\lim_{n\to\infty} B_{\rho_n}(z) = \Sigma_{\left(\mu, \nu_\boxtimes^{\gamma',\sigma'}\right)}(z)$$

uniformly on $\mathcal{D}$. Note that the function $\eta^{\langle -1\rangle}_{\nu_\boxtimes^{\gamma',\sigma'}}(z)$ can be analytically continued to the whole disk $\mathbb{D}$. It follows that there exists a unique measure $\nu \in \mathcal{M}_{\mathbb{T}}$ such that

$$B_\nu(z) = \Sigma_{\left(\mu, \nu_\boxtimes^{\gamma',\sigma'}\right)}(z), \qquad z \in \mathcal{D}.$$

Note that

$$\int_{\mathbb{T}} \zeta \, d\nu(\zeta) = B_\nu(0) = \Sigma_{\left(\mu, \nu_\boxtimes^{\gamma',\sigma'}\right)}(0) = \int_{\mathbb{T}} \zeta \, d\mu(\zeta) \neq 0.$$

We conclude that $\{\rho_n\}_{n=1}^\infty$ converges weakly to the measure $\nu \in \mathcal{M}_{\mathbb{T}}^\times$ by Proposition 5.1, and hence the limit law ν is $\boxtimes$-infinitely divisible as we have mentioned earlier. Consequently, the measure ν is of the form $\nu_\boxtimes^{\gamma,\sigma}$ for some $\gamma \in \mathbb{T}$ and a finite Borel measure σ on $\mathbb{T}$. Hence (2) holds.

Assume now (2). Then we have

$$\lim_{n\to\infty} B_{\rho_n}(z) = B_{\nu_\boxtimes^{\gamma,\sigma}}(z)$$

uniformly on compact subsets of $\mathbb{D}$. Theorem 5.5 shows that

$$\lim_{n\to\infty} \Sigma_{(\mu_n,\nu_n)}(z) = B_{\nu_\boxtimes^{\gamma,\sigma}}(z)$$

uniformly in a neighborhood of zero. Observe now there exists a measure $\mu \in \mathcal{M}_{\mathbb{T}}^\times$ such that

$$B_\mu(z) = B_{\nu_\boxtimes^{\gamma,\sigma}}\left(\eta_{\nu_\boxtimes^{\gamma',\sigma'}}(z)\right), \qquad z \in \mathbb{D}.$$

Therefore the function $B_{\nu_\boxtimes^{\gamma,\sigma}}(z)$ has the form $\Sigma_{\left(\mu, \nu_\boxtimes^{\gamma',\sigma'}\right)}(z)$ in a neighborhood of zero, and (1) follows from Proposition 5.1. $\qquad\square$

Next, we address the case that the measures μ_n converge to m, for m the Haar measure on $\mathbb{T}$.

Theorem 5.7. *Let $\{\lambda_n\}_{n=1}^\infty$, $\{\lambda_n'\}_{n=1}^\infty$ be sequences in $\mathbb{T}$, and let $\{\mu_{nk}\}_{n,k}$, $\{\nu_{nk}\}_{n,k}$ be two infinitesimal arrays in $\mathcal{M}_{\mathbb{T}}^\times$. As in the statement of* Theorem 5.6, *define*

$$(\mu_n, \nu_n) = (\delta_{\lambda_n}, \delta_{\lambda_n'}) \boxtimes (\mu_{n1}, \nu_{n1}) \boxtimes (\mu_{n2}, \nu_{n2}) \boxtimes \cdots \boxtimes (\mu_{nk_n}, \nu_{nk_n})$$

and $\rho_n = \delta_{\lambda_n} \boxtimes \mu_{n1} \boxtimes \mu_{n2} \boxtimes \cdots \boxtimes \mu_{nk_n}$, for every $n \in \mathbb{N}$, and suppose that $\{\nu_n\}_{n=1}^\infty$ converges weakly to ν. Then the following assertions are equivalent:

(1) *The sequence μ_n converges weakly to m.*
(2) *The sequence ρ_n converges weakly to m.*

Proof. Assume (1) holds. Observe that $\Sigma_{(\mu_n,\nu_n)}(0) = B_{\mu_n}(0) = \int_{\mathbb{T}} \zeta \, d\mu_n(\zeta)$ and

$$\Sigma_{(\mu_n,\nu_n)}(0) = \lambda_n \prod_{k=1}^{k_n} \Sigma_{(\mu_{nk},\nu_{nk})}(0) = \lambda_n \prod_{k=1}^{k_n} \int_{\mathbb{T}} \zeta \, d\mu_{nk}(\zeta), \qquad n \in \mathbb{N}.$$

Since the sequence μ_n converges weakly to m , the above product converges to zero as n tends to infinity. As shown in [20], Theorem 4.3, this condition implies that (2) holds.

Conversely, assume that (2) holds. Then the limit $\lim_{n\to\infty} B_{\rho_n}(z) = 0$ holds uniformly on the compact subsets of $\mathbb{D}$, and therefore Theorem 5.5 shows that

$$\lim_{n\to\infty} \Sigma_{(\mu_n,\nu_n)}(z) = 0$$

uniformly in a neighborhood of zero $\mathcal{D} \subset \mathbb{D}$. Note first that the measure ν must be $\boxtimes$-infinitely divisible (see [2]). If the measure ν has nonzero first moment, then we conclude, by Proposition 5.1, that (1) holds since the zero function is of the form $\Sigma_{(m,\nu)}$. If the first moment of the measure ν is zero, then $\nu = m$ and the sequence $\{\eta_{\nu_n}(z)\}_{n=1}^{\infty}$ converges uniformly on $\mathcal{D}$ to zero. In particular, the set $\eta_{\nu_n}(\mathcal{D})$ is contained in $\mathcal{D}$ when n is sufficiently large. We conclude in this case that the functions $B_{\mu_n}(z) = \Sigma_{(\mu_n,\nu_n)}(\eta_{\nu_n}(z))$ converge uniformly on $\mathcal{D}$ to zero as well, and therefore the sequence μ_n converges weakly to m by Proposition 5.1. □

Remark 5.8. *We remark here for a further use that the proof of Theorem 5.7 gives a convenient criterion of determining whether a limit law is the Haar measure m or not. Namely, if the free convolutions ν_n converge weakly to ν, and the c-free convolutions μ_n converge to a measure μ with $\int_{\mathbb{T}} \zeta \, d\mu(\zeta) = 0$, then the measure μ must be the Haar measure m.*

6. Infinite Divisibility

Definition 6.1. *A pair $(\mu,\nu) \in \mathcal{M}_{\mathbb{T}} \times \mathcal{M}_{\mathbb{T}}$ is said to be $\boxtimes$-infinitely divisible if for any $n \in \mathbb{N}$ there exists $(\mu_n, \nu_n) \in \mathcal{M}_{\mathbb{T}} \times \mathcal{M}_{\mathbb{T}}$ such that*

$$(\mu,\nu) = \underbrace{(\mu_n,\nu_n) \boxtimes \cdots \boxtimes (\mu_n,\nu_n)}_{n\ times}.$$

In this section, we give a complete characterization of $\boxtimes$-infinite divisibility for a pair $(\mu,\nu) \in \mathcal{M}_{\mathbb{T}} \times \mathcal{M}_{\mathbb{T}}$. We begin with the case when μ has zero first moment. We will see that these are the pairs (m,ν), where m is the Haar measure on $\mathbb{T}$ and the measure ν is $\boxtimes$-infinitely divisible.

Proposition 6.2. *Suppose that the pair (μ,ν) is $\boxtimes$-infinitely divisible and that $\int_{\mathbb{T}} \zeta \, d\mu(\zeta) = 0$. Then $\mu = m$.*

Proof. We first assume that $\int_{\mathbb{T}} \zeta \, d\nu(\zeta) \neq 0$. In this case, for every $n \in \mathbb{N}$, there exist measures $\mu_n \in \mathcal{M}_{\mathbb{T}}$ and $\nu_n \in \mathcal{M}_{\mathbb{T}}^{\times}$ such that

$$(\mu,\nu) = \underbrace{(\mu_n,\nu_n) \boxtimes \cdots \boxtimes (\mu_n,\nu_n)}_{n\ times}.$$

Thus we have $\Sigma_{(\mu,\nu)} = \left(\Sigma_{(\mu_n,\nu_n)}\right)^n$. Or, equivalently, we have

$$B_\mu(z) = \left(B_{\mu_n}\left(\eta_{\nu_n}^{\langle-1\rangle}\left(\eta_\nu(z)\right)\right)\right)^n, \qquad z \in \mathbb{D},$$

In [1], Proposition 3.3, it is shown that the function $\eta_{\nu_n}^{\langle-1\rangle}(\eta_\nu(z))$ extends analytically to the entire disk $\mathbb{D}$. Therefore the function B_μ is the n-th power of an analytic function in $\mathbb{D}$ for any $n \in \mathbb{N}$. This happens if and only if either B_μ is identically zero in $\mathbb{D}$ or $0 \notin B_\mu(\mathbb{D})$. Hence the measure μ is the $\uplus$-infinitely divisible measure with first moment zero, that is, $\mu = m$.

Suppose now $\int_{\mathbb{T}} \zeta \, d\nu(\zeta) = 0$. Then we have that

$$(\mu,\nu) = (\mu,m) = (\mu_1,m) \boxtimes (\mu_1,m)$$

for some measure $\mu_1 \in \mathcal{M}_{\mathbb{T}}$. In this case we can view the pair (μ, m) as the distribution of XY in a noncommutative C^*-probability space $(\mathcal{A}, \varphi, \psi)$, where X and Y are boolean independent random variables with the common distribution (μ_1, m). Then, for any $n \in \mathbb{N}$, the n-th moment of the measure μ is given by

$$\varphi(\underbrace{(XY) \cdots (XY)}_{n \text{ times}}) = \varphi(X)^n \varphi(Y)^n = \varphi(XY)^n = \left(\int_{\mathbb{T}} \zeta \, d\mu(\zeta) \right)^n = 0.$$

Therefore the measure μ is the Haar measure m. $\qquad\qquad\qquad\square$

We focus next on the $\boxtimes$-infinitely divisible pair (μ, m) such that $\mu \in \mathcal{M}_{\mathbb{T}}^{\times}$.

Proposition 6.3. *The above measure μ is either a point mass δ_λ for some $\lambda \in \mathbb{T}$ or the harmonic measure for the unit disc relative to some point $\alpha \in \mathbb{D} \setminus \{0\}$, i.e., $d\mu = P_\alpha \, dm$, where P_α is the Poisson kernel at α.*

Proof. By the virtue of (8), it suffices to show that the function B_μ is a constant function. Set $c = \int_{\mathbb{T}} \zeta \, d\mu(\zeta)$. Then the same argument as in the proof of Proposition 6.2 shows that the n-th moment of the measure μ is c^n. We conclude that the function

$$m_\mu(z) = cz + c^2 z^2 + \cdots = \frac{cz}{1 - cz}$$

in a neighborhood of zero. This fact implies that $B_\mu(z) = c$ for all $z \in \mathbb{D}$ as desired. $\qquad\qquad\qquad\square$

It is easy to see that if the measure ν is $\boxtimes$-infinitely divisible, a pair of measures of the form (m, ν), (δ_λ, m) or $(P_\alpha \, dm, m)$, is also $\boxtimes$-infinitely divisible.

Finally, we characterize the $\boxtimes$-infinite divisibility in the class $\mathcal{M}_{\mathbb{T}}^{\times} \times \mathcal{M}_{\mathbb{T}}^{\times}$. A family of pairs $\{(\mu_t, \nu_t)\}_{t \geq 0}$ of probability measures on $\mathbb{T}$ is a *weakly continuous semigroup* relative to the convolution $\boxtimes$ if $(\mu_t, \nu_t) \boxtimes (\mu_s, \nu_s) = (\mu_{t+s}, \nu_{t+s})$ for $t, s \geq 0$, and the maps $t \mapsto \mu_t$ and $t \mapsto \nu_t$ are continuous.

Theorem 6.4. *Given a measure $\mu \in \mathcal{M}_{\mathbb{T}}^{\times}$ and a $\boxtimes$-infinitely divisible measure $\nu \in \mathcal{M}_{\mathbb{T}}^{\times}$, the following statements are equivalent:*

(1) *The pair (μ, ν) is $\boxtimes$-infinitely divisible.*
(2) *There exists a complex number $\gamma \in \mathbb{T}$ and a finite positive Borel measure σ on $\mathbb{T}$ such that*

$$\Sigma_{(\mu, \nu)}(z) = \gamma \exp \left(- \int_{\mathbb{T}} \frac{1 + \zeta z}{1 - \zeta z} \, d\sigma(\zeta) \right)$$

in a neighborhood of zero.
(3) *There exists a weakly continuous semigroup $\{(\mu_t, \nu_t)\}_{t \geq 0}$ relative to $\boxtimes$ such that $(\mu_1, \nu_1) = (\mu, \nu)$ and $(\mu_0, \nu_0) = (\delta_1, \delta_1)$.*

Moreover, if statements (1) to (3) are all satisfied, then the limit

$$\gamma = \lim_{t \to 0^+} \exp \left(\frac{1}{t} \int_{\mathbb{T}} i \Im \zeta \, d\mu_t(\zeta) \right)$$

exists and the measure σ is the weak limit of measures

$$\frac{1}{t} (1 - \Re \zeta) \, d\mu_t(\zeta)$$

as $t \to 0^+$.

Proof. We first prove that (1) implies (2). Assume that (1) holds. For every $n \in \mathbb{N}$, we have

$$(\mu, \nu) = \underbrace{(\mu_n, \nu_n) \boxtimes \cdots \boxtimes (\mu_n, \nu_n)}_{n \, \text{times}}.$$

where $\mu_n \in \mathcal{M}_{\mathbb{T}}$ and $\nu_n \in \mathcal{M}_{\mathbb{T}}^{\times}$. The $\boxtimes$-infinite divisibility of the measure ν implies that there exists an analytic function $u(z)$ in $\mathbb{D}$ such that the function $\eta_\nu^{\langle -1 \rangle}(z) = z \exp(u(z))$ and $\Re u(z) \geq 0$ for all $z \in \mathbb{D}$ (see [4]). It follows that $\eta_{\nu_n}^{\langle -1 \rangle}(z) = z \exp(u(z)/n)$, and hence we deduce that the measures ν_n converge weakly to δ_1. On the other hand, the identity $(B_{\mu_n}(z))^n = \Sigma_{(\mu, \nu)}(\eta_{\nu_n}(z))$ and Proposition 5.1 imply that the measures μ_n converge weakly to δ_1 as well. Define two infinitesimal arrays $\{\mu_{nk}\}_{n,k}$ and $\{\nu_{nk}\}_{n,k}$ by setting $\mu_{nk} = \mu_n$ and $\nu_{nk} = \nu_n$, where $1 \leq k \leq n$. Then the measures (μ, ν) can be viewed as the weak limit of c-free convolutions $(\mu_{n1}, \mu_{n1}) \boxtimes \cdots \boxtimes (\mu_{nn}, , \nu_{nn})$. Hence (2) follows from Theorem 5.6.

Now, assume that (2) holds. It was also proved in [4] that there exists a weakly continuous semigroup $\{\nu_t\}_{t \geq 0}$ relative to $\boxtimes$ so that $\nu_0 = \delta_1$ and $\nu_1 = \nu$. Note that, for every $t \geq 0$, there exists a unique probability measure μ_t on $\mathbb{T}$ such that $B_{\mu_t}(z) = \left(\Sigma_{(\mu, \nu)}(\eta_{\nu_t}(z)) \right)^t$ for all $z \in \mathbb{D}$, where $\mu_0 = \delta_1$. Then it is easy to see that the convolution semigroup $\{(\mu_t, \nu_t)\}_{t \geq 0}$ has the desired properties.

The implication from (3) to (1) is obvious. To conclude, we only need to show the assertions about the measure σ and the number γ. Assume that the pair (μ, ν) is $\boxtimes$-infinitely divisible, and let $\{(\mu_t, \nu_t)\}_{t \geq 0}$ be the corresponding convolution semigroup as in (3). Let $\{t_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers such that $\lim_{n \to \infty} t_n = 0$. Let $k_n = [1/t_n]$ for every $n \in \mathbb{N}$, where $[x]$ denotes the largest integer that is no greater than the real number x. Observe that

$$1 - t_n < t_n k_n \leq 1, \qquad n \in \mathbb{N}.$$

Hence we have $\lim_{n \to \infty} t_n k_n = 1$, and further the properties of the semigroup $\{(\mu_t, \nu_t)\}_{t \geq 0}$ show that the c-free convolutions

$$\underbrace{(\mu_{t_n}, \nu_{t_n}) \boxtimes (\mu_{t_n}, \nu_{t_n}) \boxtimes \cdots \boxtimes (\mu_{t_n}, \nu_{t_n})}_{k_n \, \text{times}} = (\mu_{t_n k_n}, \nu_{t_n k_n})$$

converge weakly to $(\mu_1, \nu_1) = (\mu, \nu)$ as $n \to \infty$. Then Theorem 5.6 implies that the measures

$$\frac{1}{t_n}(1 - \Re \zeta) \, d\mu_{t_n}^{\circ}(\zeta) = \frac{1}{t_n k_n} k_n (1 - \Re \zeta) \, d\mu_{t_n}^{\circ}(\zeta)$$

converge weakly to the measure σ and

$$\gamma = \lim_{n \to \infty} \exp\left(\frac{1}{t_n} \int_{\mathbb{T}} i \Im \zeta \, d\mu_{t_n}^{\circ}(\zeta) \right),$$

where the centered measures $d\mu_{t_n}^{\circ}(\zeta) = d\mu_{t_n}(b_n \zeta)$ and the numbers b_n are defined as in (10). The desired result follows immediately from the fact that $\lim_{n \to \infty} b_n = 1$, and that the topology on the set $\mathcal{M}_{\mathbb{T}}$ determined by the weak convergence of measures is actually metrizable. $\qquad \square$

We conclude this section by showing a c-free analogue of Hinčin's classical theorem on the infinite divisibility of real random variables [13].

Corollary 6.5. *Let $\{\lambda_n\}_{n=1}^\infty$ and $\{\lambda'_n\}_{n=1}^\infty$ be two sequences in $\mathbb{T}$, and let $\{\mu_{nk}\}_{n,k}$ and $\{\nu_{nk}\}_{n,k}$ be two infinitesimal arrays in $\mathcal{M}_\mathbb{T}^\times$. Suppose that the sequence*

$$\left\{ (\delta_{\lambda_n}, \delta_{\lambda'_n}) \boxtimes (\mu_{n1}, \nu_{n1}) \boxtimes (\mu_{n2}, \nu_{n2}) \boxtimes \cdots \boxtimes (\mu_{nk_n}, \nu_{nk_n}) \right\}_n$$

converges weakly to (μ, ν). Then the pair (μ, ν) is $\boxtimes$-infinitely divisible.

Proof. Note first that the measure ν is $\boxtimes$-infinitely divisible as we have seen earlier. The case $\mu, \nu \in \mathcal{M}_\mathbb{T}^\times$ is an application of Theorems 5.6 and 6.4. Remark 5.8 shows that $\mu = m$ when $\int_\mathbb{T} \zeta \, d\mu(\zeta) = 0$. Only the case $\mu \in \mathcal{M}_\mathbb{T}^\times$ and $\nu = m$ requires a proof. In this case we set

$$(\mu_n, \nu_n) = (\delta_{\lambda_n}, \delta_{\lambda'_n}) \boxtimes (\mu_{n1}, \nu_{n1}) \boxtimes (\mu_{n2}, \nu_{n2}) \boxtimes \cdots \boxtimes (\mu_{nk_n}, \nu_{nk_n})$$

Observe that $B_{\mu_n}(z) = \Sigma_{(\mu_n, \nu_n)}(\eta_{\nu_n}(z))$, and that the family $\{\Sigma_{(\mu_n, \nu_n)}(z)\}_{n=1}^\infty$ is uniformly Lipschitz in a neighborhood of zero $\mathcal{D}$ with a Lipschitz constant $K > 0$. We have

$$
\begin{aligned}
\left| B_{\mu_n}(z) - \int_\mathbb{T} \zeta \, d\mu(\zeta) \right| &\leq |B_{\mu_n}(z) - B_{\mu_n}(0)| + |B_{\mu_n}(0) - B_\mu(0)| \\
&\leq K \left| \eta_{\nu_n}(z) \right| + |B_{\mu_n}(0) - B_\mu(0)|.
\end{aligned}
$$

It follows that the functions B_{μ_n} converge uniformly on compact subsets of $\mathbb{D}$ to the constant function $\int_\mathbb{T} \zeta \, d\mu(\zeta)$, and hence the function B_μ is the same constant function. The measure μ in this case is either a point mass concentrated at a point on $\mathbb{T}$ or the harmonic measure for $\mathbb{D}$ relative to a point in $\mathbb{D} \setminus \{0\}$. Therefore (μ, m) is $\boxtimes$-infinitely divisible. $\qquad\square$

7. Appendix:Non-crossing linked partitions and a formula for the coefficients of the T- and ^cT-transforms

7.1. **Non-crossing linked partitions.** The notion of non-crossing linked partitions, that we will largely use in Section 4, is a generalization of non-crossing partitions. It was first discussed in [9], in connection to the T-transform.

By a non-crossing linked partition γ of the ordered set $\{1, 2, \ldots, n\}$ we will understand a collection $B_1, \ldots, B_k$ of subsets of $\{1, 2, \ldots, n\}$, called blocks, with the following properties:

(1) $\displaystyle\bigcup_{l=1}^k B_l = \{1, \ldots, n\}$

(2) $B_1, \ldots, B_k$ are non-crossing, in the sense that there are no two blocks B_l, B_s and $i < k < p < q$ such that $i, p \in B_l$ and $k, q \in B_s$.

(3) for any $1 \leq l, s \leq k$, the intersection $B_l \bigcap B_s$ is either void or contains only one element. If $\{j\} = B_i \bigcap B_s$, then $|B_s|, |B_l| \geq 2$ and j is the minimal element of only one of the blocks B_l and B_s.

An element will be said to be *singly*, respectively *doubly* covered by γ if it is contained in exactly one, respectively exactly two blocks of γ. The set of all non-crossing linked partitions on $\{1, \ldots, n\}$ will be denoted by $NCL(n)$. If $\gamma \in NCL(n)$ and $B = i_1 < i_2 < \cdots < i_p$ is a block of γ, the element i_1 will be denoted $min(B)$. The block B will be called *exterior* if there is no other block D of γ containing two elements l, s such that $l \leq i_1 < i_p < s$. The set of all exterior blocks of γ will be denoted by $ext(\gamma)$; the set of all blocks of γ which are not exterior will be denoted by $int(\gamma)$. We will use the notation $NCL_1(n)$ for the set of all elements in $NCL(n)$ with only one exterior block.

Example: Below is represented graphically the non-crossing linked partition
$\gamma = (1, 4, 6, 9), (2, 3), (4, 5), (6, 7, 8), (10, 11), (11, 12) \in NCL(12)$:
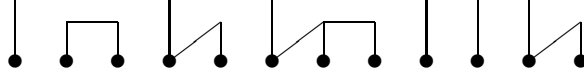
One has that $ext(\gamma) = (1, 4, 6, 9), (10, 11)$ and $int(\gamma) = (2, 3), (4, 5), (6, 7, 8), (11, 12)$.

Note that $NC(n) \subset NCL(n)$ and, for a partition in $NC(n)$, hence also in $NCL(n)$, blocks that are exterior, respectively interior, in $NC(n)$ are also in $NCL(n)$.

If $\pi \in NCL(p)$ and $\sigma \in NCL(s)$, then $\pi \oplus \sigma$ is the partition from $NCL(p + s)$ obtained by the juxtaposition of π and σ.

Finally, if $\gamma \in NCL(n)$ and A is a (ordered) subset of $\{1, 2, \ldots, n\}$ with m elements, then by $\gamma_{|A}$ we will understand the partition in $NCL(m)$ obtained by intersecting the blocks of γ with A, then shifting the elements of the ordered set A into $\{1, 2, \ldots, m\}$.

7.2. A formula for the coefficients of the T- and ^cT-transforms.

Let $X \in \mathfrak{A}$ and $m(z) = \sum_{n=1}^{\infty} m_n z^n$, $M(z) = \sum_{n=1}^{\infty} M_n z^n$ be the moment gener-ating series of X with respect to ψ, respectively φ, while $T(z) = \sum_{n=1}^{\infty} t_n z^n$ and $^cT(z) = \sum_{n=1}^{\infty} {}^c t_n z^n$ be the T-, respectively the ^cT-transform of X.

Lemma 7.1. *With the above notations, one has that:*

(i) $[T \circ (m(z))] (1 + m(z)) = \dfrac{m(z)}{z}$

(ii) $[^cT \circ (m(z))] (1 + M(z)) = \dfrac{M(z)}{z}$

Proof. (i): By definition, $T(z) = \left[\dfrac{1}{z} R(z) \right] \circ \left[R^{\langle -1 \rangle}(z) \right]$.

Since $m(z) = R(z[1 + m(z)])$, composing at left with $R^{\langle -1 \rangle}$ and at right with $m^{\langle -1 \rangle}$, we get

$$R^{\langle -1 \rangle}(z) = m^{\langle -1 \rangle}(z)(1 + z),$$

therefore

$$T(z) = \left[\frac{1}{z} R(z) \right] \circ \left[(1 + z) m^{\langle -1 \rangle}(z) \right].$$

It suffices then to show that

$$\left[\frac{1}{z} R(z) \right] \circ \left[(1 + z) m^{\langle -1 \rangle}(z) \right] \circ m(z) = \frac{m(z)}{z[1 + m(z)]}.$$

But, from (1),

$$\frac{m(z)}{z[1 + m(z)]} = \left[\frac{1}{z} R(z) \right] \circ (z[1 + m(z)])$$

and, since,

$$\left[(1+z)m^{\langle -1\rangle}(z)\right] \circ m(z) = z[1+m(z)]$$

we have q.e.d..

(ii): Analogously, it suffices to show that

$$\left[\frac{1}{z}{}^c R(z)\right] \circ \left[(1+z)m^{\langle -1\rangle}(z)\right] \circ m(z) = \frac{M(z)}{z[1+M(z)]}.$$

(2) implies

$$\frac{M(z)}{z[1+M(z)]} = \left(\frac{1}{z}{}^c R(z)\right) \circ (z[1+m(z)])$$

and, since,

$$\left[(1+z)m^{\langle -1\rangle}(z)\right] \circ m(z) = z[1+m(z)]$$

we have again q.e.d..   □

Consequence 7.2. *Let us denote, to ease the writing, $m_0 = M_0 = 1$. Then:*

(i) $m_n = \displaystyle\sum_{k=1}^{n-1} \sum_{\substack{p_1+\cdots+p_k \leq n-1 \\ p_1,\ldots,p_k \geq 0}} (t_k m_{p_1} \cdots m_{p_k})\, m_{n-1-(p_1+\cdots+p_k)}$

(ii) $M_n = \displaystyle\sum_{k=1}^{n-1} \sum_{\substack{p_1+\cdots+p_k \leq n-1 \\ p_1,\ldots,p_k \geq 0}} (t_k m_{p_1} \cdots M_{p_k})\, m_{n-1-(p_1+\cdots+p_k)}$

Theorem 7.3. *With the above notations, one has that:*

(i) $m_n = \displaystyle\sum_{\gamma \in NCL(n)} t_0^{n-|\gamma|} \cdot \prod_{B \in \gamma} t_{|B|-1}$

(ii) $M_n = \displaystyle\sum_{\gamma \in NCL(n)} t_0^{n-|\gamma|} \cdot \prod_{B \in ext(\gamma)} {}^c t_{|B|-1} \cdot \prod_{B \in int(\gamma)} t_{|B|-1}.$

Proof. Part (i) of the result is also shown in [9]. The proof presented below is shorter and employs a different approach.

First, for $\gamma \in NCL(n)$, let us denote

$$\mathcal{E}(\gamma) \;=\; t_0^{n-|\gamma|} \prod_{B \in \gamma} t_{|B|-1}$$

$$^c\mathcal{E}(\gamma) \;=\; t_0^{n-|\gamma|} \prod_{B \in ext(\gamma)} {}^c t_{|B|-1} \prod_{B \in int(\gamma)} t_{|B|-1}$$

Note that if $\gamma = \gamma_1 \oplus \gamma_2$, then $\mathcal{E}(\gamma) = \mathcal{E}(\gamma_1)\mathcal{E}(\gamma_2)$ and $^c\mathcal{E}(\gamma) = {}^c\mathcal{E}(\gamma_1)^c\mathcal{E}(\gamma_2)$.

Fix $\gamma \in NCL(n)$ and let $F = 1 < i_2 < \cdots < i_k$ be the first block of γ. Particularly, $F \in ext(\gamma)$. Let $\overline{F}$ be the smallest subset of $\{1, 2, \ldots, n\}$ with the following properties:

(i) $F \subset \overline{F}$

(ii) if $j \in \overline{F}$, then $\{1, 2, \ldots, j\} \subset \overline{F}$.

(iii) if $j \in F$ and $j \in D \in \gamma$, then $D \in \overline{F}$.

Since $\gamma = \gamma_{|\overline{F}} \oplus \gamma_{|\{1,\ldots,n\}\setminus\overline{F}}$, it follows that

$$NCL(n) = \bigsqcup_{k=1}^{n} NCL_1(k) \oplus NCL(n-k),$$

hence

$$\sum_{\gamma \in NCL(n)} \mathcal{E}(\gamma) = \sum_{k=1}^{n} \sum_{\sigma \in NCL_1(k)} \mathcal{E}(\sigma) \sum_{\pi \in NCL(n-k)} \mathcal{E}(\pi).$$

Applying the induction hypothesis, the above equality becomes

(10) $$\sum_{\gamma \in NCL(n)} \mathcal{E}(\gamma) = \sum_{k=1}^{n} \sum_{\sigma \in NCL_1(k)} \mathcal{E}(\sigma) m_{n-k}.$$

Say now $\sigma \in NCL_1(p)$ and $F = 1 < i_2 < \cdots < i_k$ is the exterior block of σ. For $2 \leq l \leq k-1$ define

$$\sigma(l) = \begin{cases} \sigma_{|\{i_l, i_l+1, \ldots, i_{l+1}-1\}} & \text{if } i_l \text{ is singly covered} \\ \sigma_{|\{i_l, i_l+1, \ldots, i_{l+1}-1\}} \setminus (i_l) & \text{if } i_l \text{ is doubly covered} \end{cases}$$
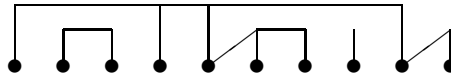
also let

$$\sigma'(k) = \begin{cases} \sigma_{|\{i_k, i_k+1, \ldots, p\}} & \text{if } i_k \text{ is singly covered} \\ \sigma_{|\{i_k, i_k+1, \ldots, p\}} \setminus (i_k) & \text{if } i_k \text{ is doubly covered} \end{cases}$$

and

$$\sigma(k) = \sigma'(k) \oplus \sigma_{|\{2,3,\ldots,i_2-1\}}$$

Example: The partition $\sigma = (1,4,5,9),(2,3),(5,6,7),(8),(9,10)$ from $NCL_1(10)$ is represented graphically below:

and we have that

$$\begin{aligned} \sigma(2) &= (1) \\ \sigma(3) &= (1,2,3)(4) \\ \sigma(4) &= (1,2)(3,4) \end{aligned}$$

Note that, since $\sigma \in NCL_1(p)$, we have that $\sigma'(k) \in NCL(p-k+1)$. Furthermore, the knowledge of $\sigma(2), \ldots, \sigma(k)$ determines σ uniquely and

$$^c\mathcal{E}(\sigma) = {}^c t_{k-1} \prod_{l=2}^{k} \mathcal{E}(\sigma(l)).$$

Combining the above result with (10) and the induction hypothesis, we get

$$
\begin{aligned}
\sum_{\gamma \in NCL(n)} {}^{c}\mathcal{E}(\gamma) &= \sum_{k=1}^{n} M_{n-k} \sum_{\sigma \in NCL_1(k)} {}^{c}\mathcal{E}(\sigma) \\
&= \sum_{k=1}^{n-1} M_{n-1-k} \sum_{\substack{p_1+\cdots+p_k \leq n-1 \\ p_1,\ldots,p_k \geq 1}} \sum_{\substack{\gamma_l \in NCL(p_l) \\ 1 \leq l \leq k}} {}^{c}t_k \prod_{l=1}^{k} (\gamma_k) \\
&= \sum_{k=1}^{n-1} M_{n-1-k} \sum_{\substack{p_1+\cdots+p_k \leq n-1 \\ p_1,\ldots,p_k \geq 1}} {}^{c}t_k \prod_{l=1}^{k} \left(\sum_{\gamma \in NCL(p_k)} \mathcal{E}(\gamma) \right) \\
&= \sum_{k=1}^{n-1} M_{n-1-k} \sum_{\substack{p_1+\cdots+p_k \leq n-1 \\ p_1,\ldots,p_k \geq 1}} {}^{c}t_k m_{p_1} \cdots m_{p_k} \\
&= M_n
\end{aligned}
$$

The inductive step for m_n is analogous (for $\varphi = \psi$, the sequences $\{m_n\}_n$ and $\{M_n\}_n$ coincide). $\qquad\square$

References

[1] S. Belinschi,H. Bercovici Partially defined semigroups relative to multiplicative free convolution, Int. Math. Res. Not. 2005, no. 2, 65-101

[2] S. Belinschi,H. Bercovici Hinčin's theorem for multiplicative free convolution, Canadian Math. Bulletin, 2008, Vol. 51, no. 1, 26-31

[3] H. Bercovici, V. Pata Stable laws and domains of attraction in free probability theory Ann. of Math., 1449 (1999), 1023–1060

[4] H. Bercovici and D. Voiculescu Lévy-Hinčin type theorems for multiplicative and additive free convolution, Pacific Journal of Mathematics, (1992), Vol. 153, No. 2, 217–248.

[5] H. Bercovici,J.C. Wang Limit theorems for free multiplicative convolutions arXiv:math/0612278

[6] F. Boca Free products of completely positive maps and spectral sets J. Funct. Anal. 97 (1991), no. 2, 251–263

[7] M. Bożejko and R. Speicher. ψ-independent and symmetrized white noises. Quantum Probability and Related Topics, (L. Accardi, ed.), World Scientific, Singapore, **VI** (1991), 219–236

[8] M. Bożejko, M. Leinert and R. Speicher. Convolution and Limit Theorems for Conditionally free Random Variables. Pac. J. Math. 175 (1996), 357-388

[9] K. Dykema. Multilinear function series and transforms in Free Probability theory. Preprint, arXiv:math.OA/0504361 v2 5 Jun 2005

[10] K. Dykema. On the S-transform over a Banach algebra Preprint, arXiv:math.OA/0501083 01/2005

[11] U. Franz Boolean convolution of probability measures on the unit circle. arXiv:math/0403243

[12] U. Haagerup. On Voiculescu's R- and S-transforms for Free non-commuting Random Variables. Fields Institute Communications, vol. 12(1997), 127–148

[13] A. Hinčin, Zur Theorie der unbeschränkt teilbaren Verteilungsgesetze Mat. Sb. 2 (1937), 79-119

[14] A.D. Krystek. Infinite divisibility for conditionally free convolution. preprint

[15] A. Nica, R. Speicher. Lectures on the Combinatorics of the Free Probability. London mathematical Society Lecture Note Series 335, Cambridge University Press 2006

[16] M. Popa. A combinatorial approach to monotonic independence over a C*-algebra. Preprint, arXiv: math.OA/0612570, 01/2007

[17] R. Speicher. Combinatorial Theory of the Free Product with amalgamation and Operator-Valued Free Probability Theory. Mem. AMS, Vol 132, No 627 (1998)

[18] M. Popa. A new proof for the multiplicative property of the boolean cumulants with applications to operator-valued case arXiv:0804.2109

[19] D.V. Voiculescu, K. Dykema, A. Nica Free random variables. CRM Monograph Series, 1. AMS, Providence, RI, 1992.

[20] J.C. Wang Limit laws for boolean convolutions, arXiv:0707.3401v2

[21] J.C. Wang Limit theorems for additive c-free convolution, (2008), preprint

INDIANA UNIVERSITY AT BLOOMINGTON, DEPARTMENT OF MATHEMATICS, RAWLES HALL, 831 E 3RD ST, BLOOMINGTON, IN 47405

E-mail address: `mipopa@indiana.edu`

INDIANA UNIVERSITY AT BLOOMINGTON, DEPARTMENT OF MATHEMATICS, RAWLES HALL, 831 E 3RD ST, BLOOMINGTON, IN 47405

E-mail address: `jiuwang@indiana.edu`